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Random-phase approximation and its extension for the O(2) anharmonic oscillator

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Abstract. We apply the random-phase approximation (RPA) and its extension called renormalized RPA to the quantum anharmonic oscillator with an $O(2)$ symmetry. We first obtain the equation for the RPA frequencies in the standard and in the renormalized RPAs using the equation-of-motion method. In the case where the ground state has a broken symmetry, we check the existence of a zero frequency in the standard and in the renormalized RPAs. Then we use a time-dependent approach where the standard-RPA frequencies are obtained as small oscillations arround the static solution in the time-dependent Hartree-Bogoliubov equation. We draw the parallel between the two approaches.

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1 Introduction

Our understanding of the low-energy behaviour of Quantum Chromodynamics calls for the developement of non-perturbative methods for quantum field theories. Some success has been obtained by adapting to relativistic quantum fields well-known non-perturbative methods used in the nuclear many-body problem. For instance, variational methods using Gaussian wave functionals (which are analogous to the Hartree-Fock-Bogoliubov (HFB) kind of approximations) have been applied to self-interacting bosonic field theory [1], and to gauge field theory [2]. One important question, which is currently intensely debated in the literature, is how do such non-perturbative approximations respect the symmetries of the theory.

In this regard, we would like in the present work to revisit this important question within the framework of yet another well-known and equally successful method from the many-body theory, the random-phase approximation (RPA) and its extension, the renormalized RPA [3,4]. The first of these two has been experienced in the past quite a number of applications in various fields either in condensed matter or in the nuclear problem. The second one, on the other hand has, since its first formulation by Rowe [5], attracted very little interest till very recently. Besides its use in the conventional nuclear problem, there has been indeed several attempts as of lately to extend its application to quantum field theoretic models as well [6,7]. It is worth reminding that in the standard RPA, two-body expectation values are evaluated in the HFB mean-field vacuum. In the renormalized RPA, however, these two-body expectation values are determined using the true RPA vacuum, inferring to the renormalized RPA a self-consistent character.

Recently, the quasi-particle RPA approach has been applied to the linear sigma model in order to obtain a correct description of the global chiral-symmetry-broken phase. The standard mean-field HFB approximation gives a finite mass to the pion. However, it has been shown that in the RPA the pion is massless in accordance with Goldstone theorem [8,9]. A careful study of the finite-temperature chiral phase transition in this model reveals, however, that the HFB-RPA approximation leads to a first-order phase transition. This clearly is an artefact of the approximation as it is well admitted that the chiral transition in this model is of second order. This problem can be traced back to the fact that the finite-temperature-induced transition does not happen in the RPA vacuum, but rather in the self-consistently built ground state, namely the HFB state. The latter is clearly a wrong vacuum for the theory as it does not possess a valley in the broken phase. The RPA fluctuations which were crucial in correcting for these shortcomings [8,9] are of no use in the present situation, since these are implemented perturbatively. Therefore, it is obvious that such an approach is

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bound to fail in describing the theory in the vicinity of the phase transition. A possible solution to this problem might very well be in relaxing the quasi-boson approximation inherent in the standard-RPA approach. This leads to a self-consistent version of the RPA very difficult to put in practice. Therefore, it is more convenient to consider rather the renormalized-RPA variant. This is what we propose to study here.

The renormalized RPA has been considered in [6] in the context of scalar $\lambda\phi^4$ field theory in 1+1 dimensions. In this work, the formulation of the RPA equations was based on the Dyson equation approach. A preliminary work applies the renormalized RPA to $O(N)$ field theory [7]. Here, we will focus on the two-dimensional quantum anharmonic oscillator with an $O(2)$ symmetry. This purely quantum-mechanical model allows to obtain analytical expressions without the problems of divergences occurring in quantum field theories. It shows the possibility to have a vacuum state with spontaneous broken symmetry. It has also the advantage that we can compare the results obtained with the standard and renormalized RPAs with exact numerical results. This numerical investigation shall be published in a future work. Here we will concentrate on the formal aspects of the theory. A crucial point is to check the existence of a zero excitation frequency above the vacuum state with broken symmetry in our non-perturbative approximations (this corresponds in quantum field theory to the Goldstone mode). Since these quantum-mechanical systems are used for demonstrational purpose, we will disregard all problems related to the infrared divergences occurring due to the presence of this zero-energy mode.

The paper is organized as follows. In the first part, we derive the renormalized-RPA equations using the equation-of-motion method [5]. In the second part, we use a time-dependent approach. In this formalism, the standard-RPA frequencies are obtained as small oscillations around the static solution in the time-dependent Hartree-Bogoliubov equation. Within this second approach, we are not able to go beyond the standard RPA. However, it is interesting to draw the parallel between the two approaches.

2 Renormalized-RPA equations from the equation-of-motion method

The Hamiltonian for the $O(2)$ anharmonic oscillator reads

$$H = \frac{P_1^2}{2} + \frac{P_2^2}{2} + \frac{\mu}{2}[X_1^2 + (\tilde{X}_2 + s)^2] + g[X_1^2 + (\tilde{X}_2 + s)^2]^2 + \eta(\tilde{X}_2 + s) \,, \tag{2.1}$$

where we have considered an explicit ($\eta \neq 0$) symmetry breaking and a spontaneous symmetry breaking along the X_2-direction: $\langle X_2 \rangle = s, X_2 = \tilde{X}_2 + s$. In the $\mu < 0$ case, the potential has a "Mexican hat" shape.

The case of two particles in a harmonic potential and coupled by a linear interaction, which is analytically solvable, has been considered by the authors of ref. [10] to demonstrate that the RPA correlation formula works well.

Let us define creation and annihilation operators $a_i^\dagger$ and $a_i, i = 1, 2$, according to

$$X_1 = \frac{1}{\sqrt{2\omega}}(a_1 + a_1^\dagger)\,, \qquad P_1 = i\sqrt{\frac{\omega}{2}}(a_1^\dagger - a_1) \,, \tag{2.2}$$

$$\tilde{X}_2 = \frac{1}{\sqrt{2\Omega}}(a_2 + a_2^\dagger)\,, \qquad P_2 = i\sqrt{\frac{\Omega}{2}}(a_2^\dagger - a_2) \,. \tag{2.3}$$

The transverse frequency ω, the radial frequency Ω and the condensate $\langle X_2 \rangle$ will be determined self-consistently. Using the analogy with the linear sigma model in quantum field theory, the X_1 and X_2 modes represent the pion and the sigma fields, respectively. In terms of the operators a_i and $a_i^\dagger$, the Hamiltonian reads

$$\begin{aligned} H = {} & \sum_{i=1,2} p_{ii} a_i^\dagger a_i + \sum_{i=1,2} p_{i0}(a_i^\dagger a_i^\dagger + a_i a_i) \\ & + g_{11}(a_1^\dagger + a_1)^4 + g_{12}(a_1^\dagger + a_1)^2(a_2^\dagger + a_2)^2 \\ & + g_{22}(a_2^\dagger + a_2)^4 + h_{12}(a_1^\dagger + a_1)^2(a_2^\dagger + a_2) \\ & + h_{22}(a_2^\dagger + a_2)^3 + \tilde{\eta}(a_2^\dagger + a_2) + C \,, \end{aligned} \tag{2.4}$$

where C is the following constant:

$$\begin{aligned} C = {} & \frac{\omega}{4} + \frac{\Omega}{4} + \frac{\mu}{4\omega} + \frac{\mu}{4\Omega} + \frac{\mu}{2}\langle X_2 \rangle^2 + \frac{3g}{\Omega}\langle X_2 \rangle^2 \\ & + \frac{g}{\omega}\langle X_2 \rangle^2 + g\langle X_2 \rangle^4 + \eta\langle X_2 \rangle \,. \end{aligned} \tag{2.5}$$

We have defined the following quantities:

$$p_{11} = \frac{\omega}{2} + \frac{\mu}{2\omega} + \frac{2g}{\omega}\langle X_2 \rangle^2 \,, \tag{2.6}$$

$$p_{22} = \frac{\Omega}{2} + \frac{\mu}{2\Omega} + \frac{6g}{\Omega}\langle X_2 \rangle^2 \,, \tag{2.7}$$

$$p_{10} = \frac{1}{2}(p_{11} - \omega)\,, \qquad p_{20} = \frac{1}{2}(p_{22} - \Omega) \,, \tag{2.8}$$

$$g_{11} = \frac{g}{4\omega^2}\,, \qquad g_{12} = \frac{g}{2\omega\Omega}\,, \qquad g_{22} = \frac{g}{4\Omega^2} \,, \tag{2.9}$$

$$h_{12} = \frac{2g}{\omega}\frac{1}{\sqrt{2\Omega}}\langle X_2 \rangle \,, \ h_{22} = \frac{2g}{\Omega}\frac{1}{\sqrt{2\Omega}}\langle X_2 \rangle \,, \tag{2.10}$$

$$\tilde{\eta} = \frac{1}{\sqrt{2\Omega}}\left(\eta + \mu\langle X_2 \rangle + 4g\langle X_2 \rangle^3\right) \,. \tag{2.11}$$

2.1 Mean-field equations

Normal ordering with respect to a_i and $a_i^\dagger$ gives the result

$$\begin{aligned} H = &\sum_{i=1,2} p_{ii} a_i^\dagger a_i + \sum_{i=1,2} p_{i0}(a_i^\dagger a_i^\dagger + a_i a_i) \\ &+g_{11}\Big\{a_1^{\dagger 4} + a_1^4 + 6a_1^\dagger a_1^\dagger a_1 a_1 + 4a_1^\dagger a_1^\dagger a_1^\dagger a_1 \\ &+4a_1^\dagger a_1 a_1 a_1 + 12 a_1^\dagger a_1 + 6a_1^\dagger a_1^\dagger + 6a_1 a_1\Big\} \\ &+g_{22}\Big\{a_2^{\dagger 4} + a_2^4 + 6a_2^\dagger a_2^\dagger a_2 a_2 + 4a_2^\dagger a_2^\dagger a_2^\dagger a_2 \\ &+4a_2^\dagger a_2 a_2 a_2 + 12 a_2^\dagger a_2 + 6a_2^\dagger a_2^\dagger + 6a_2 a_2\Big\} \\ &+g_{12}\Big\{a_1^{\dagger 2} a_2^{\dagger 2} + a_1^{\dagger 2} a_2^2 + a_1^2 a_2^{\dagger 2} + a_1^2 a_2^2 \\ &+2a_1^{\dagger 2} a_2^\dagger a_2 + 2a_1^2 a_2^\dagger a_2 + 2a_1^\dagger a_1 a_2^{\dagger 2} + 4a_1^\dagger a_1 a_2^\dagger a_2 \\ &+a_1^{\dagger 2} + a_1^2 + 2a_1^\dagger a_1 + a_2^{\dagger 2} + a_2^2 + 2a_2^\dagger a_2\Big\} \\ &+h_{12}\Big\{a_1^{\dagger 2} a_2^\dagger + a_1^{\dagger 2} a_2 + a_1^2 a_2^\dagger + a_1^2 a_2 + 2a_1^\dagger a_1 a_2^\dagger\Big\} \\ &+2a_1^\dagger a_1 a_2 + (h_{12} + 3h_{22} + \tilde{\eta})(a_2 + a_2^\dagger) + E_{\mathrm{HFB}} \ , \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} E_{\mathrm{HFB}} = &\frac{\omega}{4} + \frac{\Omega}{4} + \frac{\mu}{4\omega} + \frac{\mu}{4\Omega} \\ &+ \left(\frac{\mu}{2} + \frac{3g}{\Omega} + \frac{g}{\omega} + g\langle X_2\rangle^2\right)\langle X_2\rangle^2 \\ &+ \frac{3g}{4\omega^2} + \frac{3g}{4\Omega^2} + \frac{g}{2\omega\Omega} + \eta\left\langle \tilde{X}_2\right\rangle \end{aligned} \tag{2.13}$$

is the mean-field energy.

Minimization of E_{HFB} with respect to $\langle X_2\rangle$ gives the equation for the condensate:

$$\tilde{\eta} + 3h_{22} + h_{12} = 0\,, \tag{2.14}$$

that is

$$\eta + \langle X_2\rangle\left[\mu + 4g\langle X_2\rangle^2 + \frac{6g}{\Omega} + \frac{2g}{\omega}\right] = 0\ . \tag{2.15}$$

Minimization of E_{HFB} with respect to ω and Ω gives the two gap equations

$$\omega^2 = \mu + 4g\langle X_2\rangle^2 + \frac{6g}{\omega} + \frac{2g}{\Omega}\ , \tag{2.16}$$

$$\Omega^2 = \mu + 12g\langle X_2\rangle^2 + \frac{2g}{\omega} + \frac{6g}{\Omega}\ . \tag{2.17}$$

We check that the coefficient of the linear term in expression (2.12) gives the equation for the condensate (2.14).

Equations (2.15)-(2.17) coincide with those written by Stevenson in [11]. When there is no explicit symmetry breaking ($\eta = 0$), we have two solutions: one with $\langle X_2\rangle = 0$ and one where the symmetry is spontaneously broken, $\langle X_2\rangle \neq 0$. For the symmetric solution, we have $\omega = \Omega$ and the gap equation is

$$\omega^3 - \mu\omega - 8g = 0\ . \tag{2.18}$$

We can compare the mean-field results with exact numerical calculations [12]. For instance, for $\mu = 1$ and $g = 1$, the energies of the ground state and the two first-excited states are in the mean-field approximation: $E_0 = 1.74015, E_1 = 3.90645$ and $E_2 = 6.07275$ whereas exact numerical calculations give $E_0 = 1.7242$, $E_1 = 3.8304$ and $E_2 = 6.214$. However, we are not aware of numerical calculations for the two-dimensional anharmonic oscillator in the case $\langle X_2\rangle \neq 0$.

2.2 RPA equations from the equation-of-motion method

The symmetry generator, *i.e.* the angular-momentum operator around the 3-axis, is given by

$$L_3 = X_1 P_2 - (\tilde{X}_2 + \langle X_2\rangle)P_1\ , \tag{2.19}$$

or, in terms of the creation and annihilation operators,

$$\begin{aligned} L_3 = &\frac{i}{2}\left(\sqrt{\frac{\Omega}{\omega}} + \sqrt{\frac{\omega}{\Omega}}\right)(a_1 a_2^\dagger - a_1^\dagger a_2) \\ &-\frac{i}{2}\left(\sqrt{\frac{\Omega}{\omega}} - \sqrt{\frac{\omega}{\Omega}}\right)(a_1 a_2 - a_1^\dagger a_2^\dagger) \\ &-i\sqrt{\frac{\omega}{2}}\langle X_2\rangle(a_1^\dagger - a_1)\ . \end{aligned} \tag{2.20}$$

To derive the RPA equations, we will first use the equation-of-motion method due to Rowe [5]. We assume that an exact eigenstate $|\nu\rangle$ of the Hamiltonian can be created from the exact vacuum $|0\rangle$ by an excitation operator $Q_\nu^\dagger$:

$$|\nu\rangle = Q_\nu^\dagger|0\rangle \qquad \text{and} \qquad Q_\nu|0\rangle = 0\,. \tag{2.21}$$

Minimization of the energy $E_\nu = \langle\nu|H|\nu\rangle / \langle\nu|\nu\rangle$ with respect to a variation δQ_ν of the operator Q_ν leads to the following set of equations:

$$\left\langle 0|[\delta Q_\nu, [H, Q_\nu^\dagger]]|0\right\rangle = \Omega_\nu \left\langle 0|[\delta Q_\nu, Q_\nu^\dagger]|0\right\rangle\ , \tag{2.22}$$

where $\Omega_\nu = E_\nu - E_0$ is the excitation energy. One has also the supplementary condition

$$\langle 0|[H, Q_\nu]|0\rangle = 0\ , \tag{2.23}$$

which is equivalent to generalized mean-field equations [13].

We will restrict our choice of excitation operators $Q_\nu^\dagger$ to those which contain the same operators that appear in the symmetry generator (2.20), that is

$$\begin{aligned} Q_\nu^\dagger = &U_\nu^{(1)} a_1^\dagger - V_\nu^{(1)} a_1 + U_\nu^{(2)} a_1^\dagger a_2^\dagger \\ &-V_\nu^{(2)} a_1 a_2 + U_\nu^{(3)} a_1 a_2^\dagger - V_\nu^{(3)} a_1^\dagger a_2\ . \end{aligned} \tag{2.24}$$

The RPA ground state is defined by

$$Q_\nu |\mathrm{RPA}\rangle = 0 \ , \tag{2.25}$$

and the expectation values appearing in eqs. (2.20) and (2.23) are taken in the RPA ground state. RPA equations (2.20) become matrix equations which allow to determine the excitation energy Ω_ν and the amplitudes U_ν and V_ν:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B} & -\mathcal{A} \end{pmatrix} \begin{pmatrix} \mathcal{U}_\nu \\ \mathcal{V}_\nu \end{pmatrix} = \Omega_\nu \mathcal{N} \begin{pmatrix} \mathcal{U}_\nu \\ \mathcal{V}_\nu \end{pmatrix} \ , \tag{2.26}$$

where we have defined the three-dimensional vectors:

$$\mathcal{U}_\nu = \begin{pmatrix} U_\nu^{(1)} \\ U_\nu^{(2)} \\ U_\nu^{(3)} \end{pmatrix} , \qquad \mathcal{V}_\nu = \begin{pmatrix} V_\nu^{(1)} \\ V_\nu^{(2)} \\ V_\nu^{(3)} \end{pmatrix} . \tag{2.27}$$

The norm matrix is given by

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}_A & \mathcal{N}_B \\ \mathcal{N}_B & \mathcal{N}_A \end{pmatrix} \ , \tag{2.28}$$

where the 3×3 matrices $\mathcal{N}_A$ and $\mathcal{N}_B$ are

$$\mathcal{N}_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\tau_1+\tau_2 & \kappa_1 \\ 0 & \kappa_1 & \tau_1-\tau_2 \end{pmatrix} ,$$

$$\mathcal{N}_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\kappa_2 \\ 0 & \kappa_2 & 0 \end{pmatrix} . \tag{2.29}$$

We have introduced the notations

$$\kappa_1 = \left\langle a_1^\dagger a_1^\dagger \right\rangle = \langle a_1 a_1 \rangle \ , \quad \kappa_2 = \left\langle a_2^\dagger a_2^\dagger \right\rangle = \langle a_2 a_2 \rangle \, , \tag{2.30}$$

$$\tau_1 = \left\langle a_1^\dagger a_1 \right\rangle \ , \quad \tau_2 = \left\langle a_2^\dagger a_2 \right\rangle . \tag{2.31}$$

We have assumed that the previous quantities are real. We have also used by definition $\left\langle a_2^\dagger \right\rangle = \langle a_2 \rangle = 0$.

The RPA basis being complete, we have the following expression for any operator $\mathcal{O}$ in terms of the RPA excitation operators $Q_\nu^\dagger$ and Q_ν:

$$\mathcal{O} = \sum_{\nu=1,3} Q_\nu^\dagger \langle [Q_\nu, \mathcal{O}] \rangle - Q_\nu \langle [Q_\nu^\dagger, \mathcal{O}] \rangle \ , \tag{2.32}$$

where the expectation values are taken on the RPA ground state. We have therefore the expression of the operators $a_1^\dagger, a_1^\dagger a_2^\dagger$ and $a_1^\dagger a_2$ in terms of $Q_\nu^\dagger$ and Q_ν and we deduce

$$\left\langle a_1^\dagger \right\rangle = \left\langle a_1^\dagger a_2^\dagger \right\rangle = \left\langle a_1^\dagger a_2 \right\rangle = 0 \ . \tag{2.33}$$

We will introduce the following quantities:

$$\Gamma = \tau_1 + \kappa_1 + \tau_2 + \kappa_2 + 1 \ , \tag{2.34}$$

$$\Delta = \tau_1 + \kappa_1 - \tau_2 - \kappa_2 \ . \tag{2.35}$$

An important condition to remember in the EOM approach for RPA equations is that the norm matrix $\mathcal{N}$ has to be invertible in order to be able to write the normalization condition for the excited state $|\nu\rangle$.

The matrix elements of $\mathcal{A}$ and $\mathcal{B}$ are defined according to

$$\mathcal{A}_{11} = \left\langle [a_1, [H, a_1^\dagger]] \right\rangle \ , \tag{2.36}$$

$$\mathcal{A}_{12} = \left\langle [a_1, [H, a_1^\dagger a_2^\dagger]] \right\rangle \ , \tag{2.37}$$

$$\mathcal{A}_{13} = \left\langle [a_1, [H, a_1 a_2^\dagger]] \right\rangle \ , \tag{2.38}$$

$$\mathcal{B}_{11} = - \langle [a_1, [H, a_1]] \rangle \ , \tag{2.39}$$

$$\mathcal{B}_{12} = - \langle [a_1, [H, a_1 a_2]] \rangle \ , \tag{2.40}$$

$$\mathcal{B}_{13} = - \left\langle [a_1, [H, a_1^\dagger a_2]] \right\rangle \ , \tag{2.41}$$

$$\mathcal{A}_{21} = \left\langle [a_1 a_2, [H, a_1^\dagger]] \right\rangle \ , \tag{2.42}$$

$$\mathcal{A}_{22} = \left\langle [a_1 a_2, [H, a_1^\dagger a_2^\dagger]] \right\rangle \ , \tag{2.43}$$

$$\mathcal{A}_{23} = \left\langle [a_1 a_2, [H, a_1 a_2^\dagger]] \right\rangle \ , \tag{2.44}$$

$$\mathcal{B}_{21} = - \langle [a_1 a_2, [H, a_1]] \rangle \ , \tag{2.45}$$

$$\mathcal{B}_{22} = - \langle [a_1 a_2, [H, a_1 a_2]] \rangle \ , \tag{2.46}$$

$$\mathcal{B}_{23} = - \left\langle [a_1 a_2, [H, a_1^\dagger a_2]] \right\rangle \ , \tag{2.47}$$

$$\mathcal{A}_{31} = \left\langle [a_1^\dagger a_2, [H, a_1^\dagger]] \right\rangle \ , \tag{2.48}$$

$$\mathcal{A}_{32} = \left\langle [a_1^\dagger a_2, [H, a_1^\dagger a_2^\dagger]] \right\rangle \ , \tag{2.49}$$

$$\mathcal{A}_{33} = \left\langle [a_1^\dagger a_2, [H, a_1 a_2^\dagger]] \right\rangle \ , \tag{2.50}$$

$$\mathcal{B}_{31} = - \left\langle [a_1^\dagger a_2, [H, a_1]] \right\rangle \ , \tag{2.51}$$

$$\mathcal{B}_{32} = - \left\langle [a_1^\dagger a_2, [H, a_1 a_2]] \right\rangle \ , \tag{2.52}$$

$$\mathcal{B}_{33} = - \left\langle [a_1^\dagger a_2, [H, a_1^\dagger a_2]] \right\rangle \ . \tag{2.53}$$

We need to calculate all these commutators. In the renormalized RPA, when calculating the expectation values in the RPA ground state, one uses the following approximation:

$$\langle \alpha_i \alpha_j \alpha_k \alpha_l \rangle \simeq \langle \alpha_i \alpha_j \rangle \langle \alpha_k \alpha_l \rangle + \langle \alpha_i \alpha_k \rangle \langle \alpha_j \alpha_l \rangle + \langle \alpha_i \alpha_l \rangle \langle \alpha_j \alpha_k \rangle \ , \tag{2.54}$$

where the α operators are either creation or annihilation operators of the bosons 1 and 2. The expectation values are calculated self-consistently whereas in standard RPA, they are calculated in the HFB ground state. We obtain the following expressions for the matrix elements of $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A}_{11} \equiv \mathcal{E}_\pi = p_{11} + 12g_{11}(\Gamma+\Delta) + 2g_{12}(\Gamma-\Delta)\,, \quad (2.55)$$

$$\mathcal{A}_{12} = \tilde{\eta} + 3h_{22}(\Gamma-\Delta) + h_{12}(\Delta+3\Gamma)\,, \quad (2.56)$$

$$\mathcal{A}_{13} = 2h_{12}\Delta\,, \quad (2.57)$$

$$\mathcal{A}_{21} = 2h_{12}\Gamma\,, \quad (2.58)$$

$$\mathcal{A}_{22} = (\mathcal{E}_\pi+\mathcal{E}_\sigma)(\tau_1+\tau_2+1)+\kappa_1\chi_\pi+\kappa_2\chi_\sigma+4g_{12}\Gamma^2\,, \quad (2.59)$$

$$\mathcal{A}_{23} = -\kappa_1(\mathcal{E}_\pi-\mathcal{E}_\sigma) - (1+\tau_1+\tau_2)\chi_\pi + 4g_{12}\Gamma\Delta\,, \quad (2.60)$$

$$\mathcal{A}_{31} = 2h_{12}\Delta\,, \quad (2.61)$$

$$\mathcal{A}_{32} = \kappa_1(\mathcal{E}_\pi+\mathcal{E}_\sigma) + (\tau_1-\tau_2)\chi_\pi + 4g_{12}\Gamma\Delta\,, \quad (2.62)$$

$$\mathcal{A}_{33} = (\tau_2-\tau_1)(\mathcal{E}_\pi-\mathcal{E}_\sigma) - \kappa_1\chi_\pi - \kappa_2\chi_\sigma + 4g_{12}\Delta^2\,, \quad (2.63)$$

$$\mathcal{B}_{11} \equiv \chi_\pi = 2p_{10} + 2g_{11}(\Gamma+\Delta) + 2g_{12}(\Gamma-\Delta)\,, \quad (2.64)$$

$$\mathcal{B}_{12} = 2h_{12}\Gamma\,, \quad (2.65)$$

$$\mathcal{B}_{13} = \tilde{\eta} + h_{12}(\Gamma+3\Delta) + 3h_{22}(\Gamma-\Delta)\,, \quad (2.66)$$

$$\mathcal{B}_{21} = 2h_{12}\Gamma\,, \quad (2.67)$$

$$\mathcal{B}_{22} = \kappa_1\chi_\sigma + \kappa_2\chi_\pi + 4g_{12}\Gamma^2\,, \quad (2.68)$$

$$\mathcal{B}_{23} = -\kappa_2(\mathcal{E}_\pi-\mathcal{E}_\sigma) + (1+\tau_1+\tau_2)\chi_\sigma + 4g_{12}\Gamma\Delta\,, \quad (2.69)$$

$$\mathcal{B}_{31} = 2h_{12}\Delta\,, \quad (2.70)$$

$$\mathcal{B}_{32} = -\kappa_2(\mathcal{E}_\pi+\mathcal{E}_\sigma) + (\tau_1-\tau_2)\chi_\sigma + 4g_{12}\Gamma\Delta\,, \quad (2.71)$$

$$\mathcal{B}_{33} = \kappa_2\chi_\pi + \kappa_1\chi_\sigma + 4g_{12}\Delta^2\,. \quad (2.72)$$

The quantities $\mathcal{E}_\pi, \mathcal{E}_\sigma, \chi_\pi$ and χ_σ are defined according to

$$\mathcal{E}_\pi = \left\langle [a_1,[H,a_1^\dagger]] \right\rangle\,, \quad (2.73)$$

$$\mathcal{E}_\sigma = \left\langle [a_2,[H,a_2^\dagger]] \right\rangle\,, \quad (2.74)$$

$$\chi_\pi = -\left\langle [a_1,[H,a_1]] \right\rangle\,, \quad (2.75)$$

$$\chi_\sigma = -\left\langle [a_2,[H,a_2]] \right\rangle\,. \quad (2.76)$$

The equations for $\langle X_2\rangle$ and the frequencies ω and Ω (or for $\mathcal{E}_\pi$ and $\mathcal{E}_\sigma$) are, respectively, obtained by writing the generalized mean-field equations (2.23):

$$\langle [H,a_2] \rangle = 0\,, \quad (2.77)$$

$$\left\langle [H,a_1^\dagger a_1^\dagger] \right\rangle = 0\,, \quad (2.78)$$

$$\left\langle [H,a_2^\dagger a_2^\dagger] \right\rangle = 0\,. \quad (2.79)$$

They are equivalent to the minimization of the generalized mean-field energy $\langle H\rangle$ with respect to $\langle X_2\rangle$, ω and Ω (see the expression of $\langle H\rangle$ eq. (2.130) below). We obtain

$$\tilde{\eta} + 3h_{22}(\Gamma-\Delta) + h_{12}(\Gamma+\Delta) = 0\,, \quad (2.80)$$

$$p_{11}\kappa_1+p_{10}(2\tau_1+1)+6g_{11}(\Gamma+\Delta)^2+g_{12}(\Gamma+\Delta)(\Gamma-\Delta) = 0\,, \quad (2.81)$$

$$p_{22}\kappa_2+p_{20}(2\tau_2+1)+g_{12}(\Gamma+\Delta)(\Gamma-\Delta)+6g_{22}(\Gamma-\Delta)^2 = 0\,. \quad (2.82)$$

By using the definitions of $p_{10}, p_{20}, \mathcal{E}_\pi$ and $\mathcal{E}_\sigma$, the two last equations, which we call the generalized gap equations, can be written as

$$\mathcal{E}_\pi(\Gamma+\Delta) = \omega(2\tau_1+1)\,, \quad (2.83)$$
$$\mathcal{E}_\sigma(\Gamma-\Delta) = \Omega(2\tau_2+1)\,. \quad (2.84)$$

When using the generalized mean-field equations (2.80), (2.83) and (2.84), we check that the matrices $\mathcal{A}$ and $\mathcal{B}$ are symmetric.

2.3 Standard RPA

In the standard RPA, all expectation values are taken in the HFB ground state. We therefore have: $\tau_1 = \tau_2 = \kappa_1 = \kappa_2 = 0$ and $\Gamma = 1, \Delta = 0$. The generalized mean-field equations (2.80), (2.83) and (2.84) reduce to the mean-field equations given in sect. (2.1), eqs. (2.15), (2.16) and (2.17). We have $\mathcal{E}_\pi = \omega, \mathcal{E}_\sigma = \Omega, \chi_\pi = \chi_\sigma = 0$. The matrices $\mathcal{A}$ and $\mathcal{B}$ become much simpler:

$$\mathcal{A} = \begin{pmatrix} \omega & 2h_{12} & 0 \\ 2h_{12} & \omega+\Omega+4g_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}\,,$$
$$\mathcal{B} = \begin{pmatrix} 0 & 2h_{12} & 0 \\ 2h_{12} & 4g_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad (2.85)$$

where we have used the equation for $\langle X_2\rangle$: $\tilde{\eta}+3h_{22}+h_{12} = 0$. We note that

$$\mathcal{A} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega+\Omega & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathcal{B}\,. \quad (2.86)$$

The matrix $\mathcal{N}$ is diagonal :

$$\mathcal{N}_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \qquad \mathcal{N}_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \quad (2.87)$$

We see that the amplitudes $U_\nu^{(3)}$ and $V_\nu^{(3)}$, respectively, in front of the operators $a_1a_2^\dagger$ and $a_1^\dagger a_2$ in the excitation operator $Q_\nu^\dagger$ (2.24) decouple. In the symmetric phase, we have $L_3 = i(a_1a_2^\dagger - a_1^\dagger a_2)$ and the operators which decouple are those which appear in the symmetry generator.

The standard-RPA equations therefore reduce to a 4× 4 system (instead of 6-by-6). Its dimension will be then divided by two and the RPA frequencies Ω_ν satisfy

$$\det\left[(\mathcal{A}-\mathcal{B})(\mathcal{A}+\mathcal{B})-\Omega_\nu^2 I\right]=0\ , \tag{2.88}$$

where now $\mathcal{A}$ and $\mathcal{B}$ are 2-by-2 matrices and I is the 2-by-2 unity matrix. This gives

$$\left(\omega^2-\Omega_\nu^2\right)\left((\omega+\Omega)-\Omega_\nu^2+8g_{12}(\omega+\Omega)\right) -16h_{12}^2\omega(\omega+\Omega)=0\ . \tag{2.89}$$

This equation is valid for the two solutions $\langle X_2\rangle=0$ and $\langle X_2\rangle\neq 0$.

Let us check the existence of a zero RPA frequency in the case of a ground state with broken symmetry (the analog of the Goldstone mode in the linear sigma model). (In the following, we have no explicit symmetry breaking: $\eta=0$). By introducing the two following quantities (which correspond to loop integrals in quantum field theories):

$$I_\pi=\frac{1}{2\omega}\ ,\qquad I_\sigma=\frac{1}{2\Omega}\ , \tag{2.90}$$

the two gap equations in the case $\langle X_2\rangle\neq 0$ can be written as

$$\omega^2=8g(I_\pi-I_\sigma)\ , \tag{2.91}$$

$$\Omega^2=8g\left\langle X_2\right\rangle^2\ . \tag{2.92}$$

We then introduce the following quantity (which is analog to the self-energy in quantum field theories):

$$\Sigma(\Omega_\nu^2)=\frac{\omega+\Omega}{2\omega\Omega}\frac{1}{\Omega_\nu^2-(\omega+\Omega)^2}\ . \tag{2.93}$$

Equation (2.89) for the RPA frequencies can be rewritten as

$$\left(\Omega_\nu^2-\omega^2\right)\left(1-4g\frac{\omega+\Omega}{\omega\Omega}\frac{1}{\Omega_\nu^2-(\omega+\Omega)^2}\right) -32g^2\left\langle X_2\right\rangle^2\frac{\omega+\Omega}{\omega\Omega}\frac{1}{\Omega_\nu^2-(\omega+\Omega)^2}=0\ , \tag{2.94}$$

or

$$\Omega_\nu^2-\omega^2=64g^2\left\langle X_2\right\rangle^2\frac{\Sigma(\Omega_\nu^2)}{1-8g\Sigma(\Omega_\nu^2)}\ . \tag{2.95}$$

This equation is identical to eq. (28) of [6]. We also notice that

$$\Sigma(\Omega_\nu=0)=-\frac{1}{2\omega\Omega(\omega+\Omega)}\ , \tag{2.96}$$

$$I_\pi-I_\sigma=\frac{\Omega^2-\omega^2}{2\omega\Omega(\omega+\Omega)}\ . \tag{2.97}$$

Therefore,

$$\Sigma(\Omega_\nu=0)=\frac{I_\pi-I_\sigma}{\omega^2-\Omega^2}\ . \tag{2.98}$$

We then use the gap equation (2.92) to write the equation for the RPA frequencies (2.95) as

$$\Omega_\nu^2=\omega^2+8g\Omega^2\frac{\Sigma(\Omega_\nu^2)}{1-8g\Sigma(\Omega_\nu^2)}\ , \tag{2.99}$$

and then we use the first gap equation (2.91) to obtain

$$\Omega_\nu^2=\frac{8g}{1-8g\Sigma(\Omega_\nu^2)}(\Omega^2-\omega^2)\left(\Sigma(\Omega_\nu^2)-\Sigma(\Omega_\nu^2=0)\right)\ . \tag{2.100}$$

It is clear now that there is a zero frequency in the RPA spectrum.

2.4 Renormalized RPA

We see from eqs. (2.29) and (2.30) that if κ_1 or κ_2 are not vanishing, the norm matrix $\mathcal{N}$ is not diagonal. In this case, it will not always be possible to write normalization conditions for the excited states $|\nu\rangle$. We therefore impose the supplementary conditions $\left\langle a_1^\dagger a_1^\dagger\right\rangle=\langle a_1a_1\rangle=0$ and $\left\langle a_2^\dagger a_2^\dagger\right\rangle=\langle a_2a_2\rangle=0$, that is $\kappa_1=0$ and $\kappa_2=0$, *i.e.* we neglect pair correlations. With these conditions, the matrix $\mathcal{N}$ stays diagonal in the renormalized approximation

$$\mathcal{N}=\begin{pmatrix}\mathcal{N}_A & 0\\ 0 & \mathcal{N}_A\end{pmatrix}\quad\text{with}$$

$$\mathcal{N}_A=\begin{pmatrix}1 & 0 & 0\\ 0 & 1+\tau_1+\tau_2 & 0\\ 0 & 0 & \tau_1-\tau_2\end{pmatrix}\ . \tag{2.101}$$

The matrices $\mathcal{A}$ and $\mathcal{B}$ simplify:

$$\mathcal{A}=\begin{pmatrix}\omega & 0 & 0\\ 0 & (\omega+\Omega)(1+\tau_1+\tau_2) & 0\\ 0 & 0 & (\Omega-\omega)(\tau_1-\tau_2)\end{pmatrix}+\mathcal{B}\ , \tag{2.102}$$

$$\mathcal{B}=\begin{pmatrix}0 & 2h_{12}\Gamma & 2h_{12}\Delta\\ 2h_{12}\Gamma & 4g_{12}\Gamma^2 & 4g_{12}\Gamma\Delta\\ 2h_{12}\Delta & 4g_{12}\Gamma\Delta & 4g_{12}\Delta^2\end{pmatrix}\ . \tag{2.103}$$

We notice that, contrary to what happen in standard RPA, the operators $a_1a_2^\dagger$ and $a_1^\dagger a_2$ do not decouple: the RPA matrix remains 6-by-6. Its expression is very similar to standard RPA at finite temperature, τ_1 and τ_2 being the occupation numbers [9].

We defined new three-dimensional vectors $\bar{\mathcal{U}}_\nu$ and $\bar{\mathcal{V}}_\nu$ and a new RPA matrix $\bar{\mathcal{R}}$ by

$$\begin{pmatrix}\bar{\mathcal{U}}_\nu\\ \bar{\mathcal{V}}_\nu\end{pmatrix}=\mathcal{N}^{1/2}\begin{pmatrix}\mathcal{U}_\nu\\ \mathcal{V}_\nu\end{pmatrix}\ ,\quad \bar{\mathcal{R}}=\mathcal{N}^{-1/2}\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ -\mathcal{B} & -\mathcal{A}\end{pmatrix}\mathcal{N}^{-1/2}\ . \tag{2.104}$$

We have

$$\bar{\mathcal{R}}=\begin{pmatrix}\bar{\mathcal{A}} & \bar{\mathcal{B}}\\ -\bar{\mathcal{B}} & -\bar{\mathcal{A}}\end{pmatrix}\ , \tag{2.105}$$

with $\bar{\mathcal{A}}=\mathcal{N}_A^{-1/2}\mathcal{A}\mathcal{N}_A^{-1/2}$ and $\bar{\mathcal{B}}=\mathcal{N}_A^{-1/2}\mathcal{B}\mathcal{N}_A^{-1/2}$. The renormalized-RPA equations then write

$$\begin{pmatrix}\bar{\mathcal{A}} & \bar{\mathcal{B}}\\ -\bar{\mathcal{B}} & -\bar{\mathcal{A}}\end{pmatrix}\begin{pmatrix}\bar{\mathcal{U}}_\nu\\ \bar{\mathcal{V}}_\nu\end{pmatrix}=\Omega_\nu\begin{pmatrix}\bar{\mathcal{U}}_\nu\\ \bar{\mathcal{V}}_\nu\end{pmatrix}\ , \tag{2.106}$$

and the RPA frequencies are determined by

$$\det\left[(\bar{\mathcal{A}}-\bar{\mathcal{B}})(\bar{\mathcal{A}}+\bar{\mathcal{B}})-\Omega_\nu^2 I\right]=0\ , \tag{2.107}$$

I being the 3-by-3 unity matrix. The matrices $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ are given by

$$\bar{\mathcal{A}} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega+\Omega & 0 \\ 0 & 0 & \Omega-\omega \end{pmatrix} + \bar{\mathcal{B}} \,, \tag{2.108}$$

$$\bar{\mathcal{B}} = \begin{pmatrix} 0 & 2h_{12}\sqrt{\Gamma} & 2h_{12}\sqrt{\Delta} \\ 2h_{12}\sqrt{\Gamma} & 4g_{12}\Gamma & 4g_{12}\sqrt{\Gamma\Delta} \\ 2h_{12}\sqrt{\Delta} & 4g_{12}\sqrt{\Gamma\Delta} & 4g_{12}\Delta \end{pmatrix} \,, \tag{2.109}$$

where $\Gamma = 1+\tau_1+\tau_2$ and $\Delta = \tau_1-\tau_2$.

The norm of the excited states

$$\langle\nu|\nu\rangle = \langle 0|Q_\nu Q_\nu^\dagger|0\rangle = \langle 0|[Q_\nu, Q_\nu^\dagger]|0\rangle \tag{2.110}$$

can be chosen to be equal to one. This corresponds to

$$(\mathcal{U}_\nu \; \mathcal{V}_\nu) \begin{pmatrix} \mathcal{N}_A & 0 \\ 0 & -\mathcal{N}_A \end{pmatrix} \begin{pmatrix} \mathcal{U}_\nu \\ \mathcal{V}_\nu \end{pmatrix} = 1 \,, \tag{2.111}$$

or

$$(\bar{\mathcal{U}}_\nu \; \bar{\mathcal{V}}_\nu) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \bar{\mathcal{U}}_\nu \\ \bar{\mathcal{V}}_\nu \end{pmatrix} = 1 \,, \tag{2.112}$$

i.e.

$$(\bar{U}_\nu^{(1)})^2 + (\bar{U}_\nu^{(2)})^2 + (\bar{U}_\nu^{(3)})^2 - (\bar{V}_\nu^{(1)})^2 - (\bar{V}_\nu^{(2)})^2 - (\bar{V}_\nu^{(3)})^2 = 1 \,. \tag{2.113}$$

By replacing the expressions for g_{12} and h_{12}, eq. (2.107) can be written in the form

$$(\omega^2-\Omega_\nu^2)\left[1-4g\left(\frac{\omega+\Omega}{\omega\Omega}\frac{\Gamma}{\Omega_\nu^2-(\omega+\Omega)^2} - \frac{\omega-\Omega}{\omega\Omega}\frac{\Delta}{\Omega_\nu^2-(\omega-\Omega)^2}\right)\right] = -32g^2\langle X_2\rangle^2\left[-\frac{\omega-\Omega}{\omega\Omega}\frac{\Delta}{\Omega_\nu^2-(\omega-\Omega)^2} + \frac{\omega+\Omega}{\omega\Omega}\frac{\Gamma}{\Omega_\nu^2-(\omega+\Omega)^2}\right]. \tag{2.114}$$

We introduce the quantity

$$\Sigma_{\rm r}(\Omega_\nu^2) = \frac{\omega+\Omega}{2\omega\Omega}\frac{1+\tau_1+\tau_2}{\Omega_\nu^2-(\omega+\Omega)^2} - \frac{\omega-\Omega}{2\omega\Omega}\frac{\tau_1-\tau_2}{\Omega_\nu^2-(\omega-\Omega)^2} \,, \tag{2.115}$$

which is formally similar to the self-energy operator in quantum field theories at finite temperature. Equation (2.114) then becomes

$$\Omega_\nu^2 = \omega^2 + 64g^2\langle X_2\rangle^2 \frac{\Sigma_{\rm r}(\Omega_\nu^2)}{1-8g\Sigma_{\rm r}(\Omega_\nu^2)} \,. \tag{2.116}$$

This equation for the RPA frequencies Ω_ν has the same form as in standard RPA (eq. (2.95)) but, in renormalized RPA, $\Sigma_{\rm r}$ contains the densities τ_1 and τ_2 which have to be determined self-consistently.

Proceeding in the same way as in standard RPA, let us check the existence of a zero frequency in renormalized RPA in the case $\langle X_2\rangle \neq 0$. We first introduce $I_\pi = (2\tau_1+1)/2\omega$ and $I_\sigma = (2\tau_2+1)/2\Omega$. The equation for the condensate and the two generalized gap equations have then the same form as in standard RPA:

$$\mu + 4g\langle X_2\rangle^2 + 4gI_\pi + 12gI_\sigma = 0 \,, \tag{2.117}$$

$$\omega^2 = \mu + 4g\langle X_2\rangle^2 + 12gI_\pi + 4gI_\sigma \,, \tag{2.118}$$

$$\Omega^2 = \mu + 12g\langle X_2\rangle^2 + 4gI_\pi + 12I_\sigma \,, \tag{2.119}$$

and we have again

$$\omega^2 = 8g(I_\pi - I_\sigma) \,, \tag{2.120}$$

$$\Omega^2 = 8g\langle X_2\rangle^2 \,. \tag{2.121}$$

We have

$$\Sigma_{\rm r}(\Omega_\nu^2=0) = -\frac{1}{2\omega\Omega}\frac{1+\tau_1+\tau_2}{(\omega+\Omega)} + \frac{1}{2\omega\Omega}\frac{\tau_1-\tau_2}{(\omega-\Omega)} \tag{2.122}$$

or

$$\Sigma_{\rm r}(\Omega_\nu^2=0) = \frac{I_\pi - I_\sigma}{\omega^2-\Omega^2} \,. \tag{2.123}$$

Using the two eqs. (2.120) and (2.121), we obtain the same expression as in standard RPA with Σ replaced by $\Sigma_{\rm r}$:

$$\Omega_\nu^2 = \frac{8g}{1-8g\Sigma_{\rm r}(\Omega_\nu^2)}(\Omega^2-\omega^2)\left(\Sigma_{\rm r}(\Omega_\nu^2) - \Sigma_{\rm r}(\Omega_\nu^2=0)\right) \,. \tag{2.124}$$

We have therefore proven the existence of a zero frequency in the RPA spectrum in the case where we have a broken symmetry in standard RPA as well as in renormalized RPA. This result is very encouraging to apply renormalized RPA in the linear sigma model.

The operator associated to the zero mode is Hermitian and cannot be normalized according to (2.113). It is equal to the symmetry generator L_3. The amplitudes $\bar{\mathcal{U}}_0$ and $\bar{\mathcal{V}}_0$ for the zero mode are

$$\bar{U}_0^{(1)} = -i\sqrt{\frac{\omega}{2}}\langle X_2\rangle \,, \tag{2.125}$$

$$\bar{U}_0^{(2)} = \frac{i}{2}\left(\sqrt{\frac{\Omega}{\omega}} - \sqrt{\frac{\omega}{\Omega}}\right)\sqrt{\Gamma} \,, \tag{2.126}$$

$$\bar{U}_0^{(3)} = \frac{i}{2}\left(\sqrt{\frac{\Omega}{\omega}} + \sqrt{\frac{\omega}{\Omega}}\right)\sqrt{\Delta} \,, \tag{2.127}$$

and $\bar{\mathcal{U}}_0 = \bar{\mathcal{V}}_0$.

In order to obtain a closing of the renormalized-RPA eigenvalues problem, one needs the expressions of the expectation values τ_1 and τ_2 in terms of the RPA amplitudes. One uses the inversion formula (2.32) and the algebra of the $sp(4)$ group:

$$a_1^\dagger a_1 = \frac{1}{2}\left([a_1a_2, a_1^\dagger a_2^\dagger] + [a_1^\dagger a_2, a_1 a_2^\dagger] - 1\right) \,, \tag{2.128}$$

$$a_2^\dagger a_2 = \frac{1}{2}\left([a_1a_2, a_1^\dagger a_2^\dagger] - [a_1^\dagger a_2, a_1 a_2^\dagger] - 1\right) \,. \tag{2.129}$$

This provides a set of supplementary equations and allows a closing of the renormalized-RPA eigenvalues problem.

To be complete, we finally give the expression of the energy in the renormalized RPA. From the expression of H with the normal ordered product (2.12), we obtain

$$\begin{aligned}\langle H\rangle = & p_{11}(\tau_1+\kappa_1)+p_{22}(\tau_2+\kappa_2)-\omega\kappa_1-\Omega\kappa_2 \\ & +12g_{11}(\tau_1+\kappa_1)(\tau_1+\kappa_1+1) \\ & +12g_{22}(\tau_2+\kappa_2)(\tau_2+\kappa_2+1) \\ & +2g_{12}(2(\tau_1+\kappa_1)(\tau_2+\kappa_2) \\ & +\tau_1+\kappa_1+\tau_2+\kappa_2)+E_{\mathrm{HFB}}\,. \end{aligned} \tag{2.130}$$

We have checked that minimization of $\langle H\rangle$ with respect to $\langle X_2\rangle, \omega$ and Ω gives eq. (2.80) for $\langle X_2\rangle$ and the two generalized gap equations (2.81) and (2.82).

2.5 Renormalized RPA for the symmetric solution

For the symmetric solution $\langle X_2\rangle = 0$, we have $\omega = \Omega, \tau_1 = \tau_2$. We still use $\kappa_1 = \kappa_2 = 0$.

The matrix $\mathcal{N}$ is diagonal with a vanishing matrix $\mathcal{N}_B$ and

$$\mathcal{N}_A = \begin{pmatrix}1 & 0 & 0\\ 0 & 1+2\tau_1 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{2.131}$$

The renormalised RPA matrix writes

$$\bar{\mathcal{A}} = \begin{pmatrix}\omega & 0 & 0\\ 0 & 2\omega & 0\\ 0 & 0 & 0\end{pmatrix} + \bar{\mathcal{B}} \quad \text{with} \quad \bar{\mathcal{B}} = \begin{pmatrix}0 & 0 & 0\\ 0 & \frac{2g}{\omega^2}(1+2\tau_1) & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{2.132}$$

For the symmetric solution we therefore obtain a decoupling of the amplitudes in front of the operators $a_1 a_2^\dagger$ and $a_1^\dagger a_2$ in renormalized RPA, contrary to what happens for the solution with broken symmetry.

The RPA frequencies are the solutions of

$$\left(\Omega_\nu - \omega^2\right)\left[\Omega_\nu^2 - 4\omega^2 - \frac{8g}{\omega}(1+2\tau_1)\right] = 0 \tag{2.133}$$

and the generalized gap equation writes

$$\omega^2 = \mu + \frac{8g}{\omega}(2\tau_1+1)\,. \tag{2.134}$$

We have therefore the RPA frequencies $\Omega_\nu^2 = \omega^2$ and $\Omega_\nu^2 = 5\omega^2 - \mu$.

3 RPA from the time-dependent formalism

In this second part of our paper, we will derive the RPA frequencies from the linearization of the time-dependent Hartree-Bogoliubov (TDHB) equations. This approach has been introduced in many-body non-relativistic theories [14]. It has also been used in $\lambda\Phi^4$ field theory in ref. [15], where small oscillations in the broken phase lead to one- and two-meson modes of the theory. In this formalism, the natural variables are $\langle X_i\rangle$ and $\langle P_i\rangle$ (or $\Phi(\vec{x})$ and $\Pi(\vec{x})$ in $\lambda\Phi^4$ field theory [15]). This formalism is well adapted to dynamical problems. Some people working on RPAs use the formalism with the creation and annihilation operators, other use the time-dependent variational approach. It is interesting to make a close comparison between the two approaches.

For a two-dimensional system, a Gaussian state at finite temperature can be described by a vector α^a and a matrix Ξ^{ab}, $a = 1, 2$:

$$\alpha^a = \begin{pmatrix}\bar{x}^a\\ -i\bar{p}^a\end{pmatrix}, \qquad \Xi^{ab} = \begin{pmatrix}2G^{ab} & -iT^{ab}\\ -iT^{ba} & -2S^{ab}\end{pmatrix} \tag{3.1}$$

with

$$\bar{x}^a = \langle X^a\rangle\,, \qquad \bar{p}^a = \langle P^a\rangle\,, \tag{3.2}$$

$$G^{ab} = \left\langle \tilde{X}^a \tilde{X}^b\right\rangle, \tag{3.3}$$

$$S^{ab} = \left\langle \tilde{P}^a \tilde{P}^b\right\rangle, \tag{3.4}$$

$$T^{ab} = \left\langle \tilde{X}^a \tilde{P}^b + \tilde{P}^b \tilde{X}^a\right\rangle \tag{3.5}$$

with $\tilde{X}^a = X^a - \langle X^a\rangle$ and $\tilde{P}^a = P^a - \langle P^a\rangle$. The matrices G and S are symmetric.

At zero temperature, we have only two independent matrices among G, T and S. The state can be described by a Gaussian wave function parameterized by $\vec{\bar{x}}, \vec{\bar{p}}, G$ and Σ:

$$\begin{aligned}\psi(X^1, X^2, t) = & \frac{1}{\mathcal{N}}\exp\left(-\left\langle \vec{X}-\vec{\bar{x}}\right| \frac{1}{4G} + i\Sigma \left|\vec{X}-\vec{\bar{x}}\right\rangle\right) \\ & \times \exp\left(i\left\langle \vec{\bar{p}} | \vec{X}-\vec{\bar{x}}\right\rangle\right), \end{aligned} \tag{3.6}$$

where the matrix Σ is related to the preceding matrices by

$$T^{ab} = 2(G\Sigma + \Sigma G)^{ab}\,, \tag{3.7}$$

$$S^{ab} = \frac{1}{4}(G^{-1})^{ab} + 4(\Sigma G \Sigma)^{ab}\,. \tag{3.8}$$

If we work with the operators $a_i, a_i^\dagger$, as is more usual in many-body problems, one introduces the matrix ρ defined by

$$(1+2\rho)^{ij} = \begin{pmatrix}\left\langle \tilde{a}_i\tilde{a}_j^\dagger + \tilde{a}_j^\dagger\tilde{a}_i\right\rangle & -2\left\langle \tilde{a}_i\tilde{a}_j\right\rangle\\ 2\left\langle \tilde{a}_i^\dagger\tilde{a}_j^\dagger\right\rangle & -\left\langle \tilde{a}_i^\dagger\tilde{a}_j + \tilde{a}_j\tilde{a}_i^\dagger\right\rangle\end{pmatrix}. \tag{3.9}$$

The link with the representation with operators $a_i, a_i^\dagger$ and the representation with X_i, P_i is given by the matrix relation

$$\Xi' = \frac{1}{2}(1+\tau)(1+2\rho)(1+\tau)\,, \tag{3.10}$$

where

$$\tau = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}, \tag{3.11}$$

I being the 2-by-2 matrix and

see eq. (3.12) *on the next page*

with $\omega_1 = \omega$ and $\omega_2 = \Omega$ (see eqs. (2.2), (2.3)).

$$\Xi' = \begin{pmatrix} 2\sqrt{\omega_i}\left\langle \tilde{X}_i\tilde{X}_j \right\rangle \sqrt{\omega_j} & -i\sqrt{\omega_i}\left\langle \tilde{X}_i\tilde{P}_j + \tilde{P}_j\tilde{X}_i \right\rangle \frac{1}{\sqrt{\omega_j}} \\ -i\sqrt{\omega_j}\left\langle \tilde{X}_j\tilde{P}_i + \tilde{P}_i\tilde{X}_j \right\rangle \frac{1}{\sqrt{\omega_i}} & -2\frac{1}{\sqrt{\omega_i}}\left\langle \tilde{P}_i\tilde{P}_j \right\rangle \frac{1}{\sqrt{\omega_j}} \end{pmatrix} \quad (3.12)$$

3.1 The time-dependent Hartree-Bogoliubov equations

In this part of the paper, we will use the following notation for the Hamiltonian of two-dimensional quantum anharmonic oscillator:

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + g(X_1^2 + X_2^2 - a^2)^2 \ . \quad (3.13)$$

The time-dependent variational Hartree-Bogoliubov equations are obtained at zero temperature from the minimization of

$$S = \int \mathrm{d}t \ \langle\psi|i\partial_t - H|\psi\rangle \ , \quad (3.14)$$

where $|\psi\rangle$ is the state corresponding to the variational wave function (3.6). At finite temperature, the TDHB equations are obtained by minimizing [16]

$$\mathcal{Z}(\mathcal{D}(t)) = \mathrm{tr}(\mathcal{D}(t_1)) - \int_{t_0}^{t_1} \mathrm{d}t\,\mathrm{tr}\left(\frac{\mathrm{d}\mathcal{D}(t)}{\mathrm{d}t} + i[H,\mathcal{D}(t)]\right) \ , \quad (3.15)$$

where $\mathcal{D}(t)$ is a variational density matrix, chosen to be a Gaussian and therefore characterized by α and Ξ.

The TDHB equations can be written in the following compact form [17]:

$$i\dot{\alpha} = \tau w \ , \quad (3.16)$$

$$i\dot{\Xi} = -\left[(\Xi+\tau)\,\mathcal{H}\,(\Xi-\tau) - (\Xi-\tau)\,\mathcal{H}\,(\Xi+\tau)\right] \ , \quad (3.17)$$

or

$$i\dot{\Xi} = 2\left[\Xi\,\mathcal{H}\,\tau - \tau\,\mathcal{H}\,\Xi\right] \ . \quad (3.18)$$

The vector w and the matrix $\mathcal{H}$ are defined by

$$\delta\langle H\rangle = \tilde{w}_i^a\delta\alpha_i^a - \frac{1}{2}\mathrm{tr}\left(\mathcal{H}_{ij}^{ab}\delta\Xi_{ji}^{ba}\right) \ . \quad (3.19)$$

For the Hamiltonian (3.13), we have

$$\begin{aligned}\langle H\rangle = &\frac{1}{2m}(p_1^2 + p_2^2 + \mathrm{tr}S) \\ &+gG_{11}(6\bar{x}_1^2 + 2\bar{x}_2^2 - 2a^2 + 3G_{11} + G_{22}) \\ &+gG_{22}(2\bar{x}_1^2 + 6\bar{x}_2^2 - 2a^2 + G_{11} + 3G_{22}) \\ &+gG_{12}(8\bar{x}_1\bar{x}_2 + 4G_{12}) + g(\bar{x}_1^2 + \bar{x}_2^2 - a^2)^2. \end{aligned} \quad (3.20)$$

The vector w and the matrix $\mathcal{H}$ are given by

$$w_2^1 = \frac{i}{m}\bar{p}_1 \,, \qquad w_2^2 = \frac{i}{m}\bar{p}_2 \,, \quad (3.21)$$

$$\begin{aligned} w_1^1 = &4g\bar{x}_1\left(\bar{x}_1^2 + \bar{x}_2^2 - a^2\right) + 12g\bar{x}_1G^{11} \\ &+4g\bar{x}_1G^{22} + 8g\bar{x}_2G^{12} \,, \end{aligned} \quad (3.22)$$

$$\begin{aligned} w_1^2 = &4g\bar{x}_2\left(\bar{x}_1^2 + \bar{x}_2^2 - a^2\right) + 12g\bar{x}_2G^{22} \\ &+4g\bar{x}_2G^{22} + 8g\bar{x}_1G^{12} \,, \end{aligned} \quad (3.23)$$

$$\mathcal{H}_{11}^{11} = -6g\bar{x}_1^2 - 2g\bar{x}_2^2 - 6gG^{11} - 2gG^{22} + 2ga^2 \,, \quad (3.24)$$

$$\mathcal{H}_{11}^{12} = -4g\bar{x}_1\bar{x}_2 - 4gG^{12} = \mathcal{H}_{11}^{21} \,, \quad (3.25)$$

$$\mathcal{H}_{11}^{22} = -6g\bar{x}_2^2 - 2g\bar{x}_1^2 - 6gG^{22} - 2gG^{11} + 2ga^2 \,, \quad (3.26)$$

$$\mathcal{H}_{22}^{ab} \equiv \frac{\delta\langle H\rangle}{\delta S^{ba}} = \frac{1}{2m}\delta^{ab} \,, \quad (3.27)$$

$$\mathcal{H}_{12}^{ab} \equiv 2i\frac{\delta\langle H\rangle}{\delta T^{ab}} = 0 \,, \qquad \mathcal{H}_{21}^{ab} \equiv 2i\frac{\delta\langle H\rangle}{\delta T^{ba}} = 0 \,. \quad (3.28)$$

3.2 Static solution of the TDHB equations

The static solution of the TDHB equations (3.16) and (3.18) is given by

$$\bar{w} = 0 \,, \quad (3.29)$$

$$\bar{\Xi}\,\mathcal{H}(\bar{\alpha},\bar{\Xi})\,\tau - \tau\,\mathcal{H}(\bar{\alpha},\bar{\Xi})\,\bar{\Xi} = 0 \,. \quad (3.30)$$

Using the rotational invariance we can choose $\bar{x}^1 = 0$ and from $\bar{w} = 0$ we deduce also $G^{12} = 0$ (to simplify the notations we do not use the bar on the matrix elements of Ξ for the HFB ground state). From $\bar{w} = 0$, we deduce the existence of two solutions. One solution is symmetric with $\bar{x}_2 = 0$. The other solution shows a broken symmetry with $\bar{x}_2 \neq 0$ and given by

$$\bar{x}_2^2 = a^2 - G^{11} - 3G^{22} \,. \quad (3.31)$$

From eq. (3.30), we deduce

$$T^{ab} = 0 \,, \quad (3.32)$$

$$S^{12} = 0 \,, \quad (3.33)$$

$$S^{11} = 4mgG^{11}(\bar{x}_2^2 - a^2 + 3G^{11} + G^{22}) \,, \quad (3.34)$$

$$S^{22} = 4mgG^{22}(3\bar{x}_2^2 - a^2 + G^{11} + G^{22}) \,. \quad (3.35)$$

For the symmetric solution, we have

$$S^{11} = S^{22} = 4mg(4G^{11} - a^2) \ . \quad (3.36)$$

For the solution with broken symmetry,

$$S^{11} = 8gm(G^{11} - G^{22})G^{11} \,, \quad (3.37)$$

$$S^{22} = 8gmG^{22}\bar{x}_2^2 \,. \quad (3.38)$$

At zero temperature, we have for the static solution: $T^{ab} = 0$ and $S^{11} = 1/4G^{11}, S^{22} = 1/4G^{22}$. Equations (3.34) and (3.35) give the two gap equations

$$\frac{1}{8m}G_{11}^{-2} = 2g(\bar{x}_2^2 - a^2 + 3G_{11} + G_{22}) \,, \quad (3.39)$$

$$\frac{1}{8m}G_{22}^{-2} = 2g(3\bar{x}_2^2 - a^2 + G_{11} + G_{22}) \,. \quad (3.40)$$

By using the identification $G^{11} = 1/2\omega$ and $G^{22} = 1/2\Omega$, the previous equations are identical to the mean-field gap equations obtained in the first part of the paper, (2.16) and (2.17).

At zero temperature, the symmetric phase is characterized by

$$\bar{x}_2 = 0\,, \tag{3.41}$$

$$G_{11} = G_{22}\,, \qquad \frac{1}{8m}G_{11}^{-2} = 2g(4G_{11} - a^2)\ . \tag{3.42}$$

The solution with broken symmetry at zero temperature is characterized by

$$\bar{x}_2^2 = a^2 - G_{11} - 3G_{22}\,, \tag{3.43}$$

$$\frac{1}{32m}G_{11}^{-2} = g(G_{11} - G_{22})\,, \tag{3.44}$$

$$\frac{1}{32m}G_{22}^{-2} = g\bar{x}_2^2\,. \tag{3.45}$$

3.3 Small oscillations around the static solution

The linearization of the TDHB equations (3.16) and (3.18) writes

$$i\delta\dot{\alpha} = \delta w\,, \tag{3.46}$$

$$i\delta\dot{\Xi} = 2\left[\delta\Xi\,\bar{\mathcal{H}}\,\tau - \tau\,\bar{\mathcal{H}}\,\delta\Xi + \bar{\Xi}\,\delta\mathcal{H}\,\tau - \tau\,\delta\mathcal{H}\,\bar{\Xi}\right]\ , \tag{3.47}$$

where $\bar{\mathcal{H}}$ is the matrix $\mathcal{H}$ evaluated for the HFB static solution $\bar{\alpha}, \bar{\Xi}$.

Let us write more explicitly the linearization of the TDHB equations around the static HFB solution with *broken symmetry* characterized by $\bar{x}_1 = 0, \bar{x}_2 \neq 0$, $G^{11}, G^{22}, S^{11}, S^{22}, \bar{p}_1 = \bar{p}_2 = 0, G^{12} = S^{12} = T^{ab} = 0$. We obtain a differential system with 14 variables: $\delta x_1, \delta p_1, \delta x_2, \delta p_2, \delta G^{11}, \delta G^{12}, \delta G^{22}, \delta T^{11}, \delta T^{12}, \delta T^{21}, \delta T^{22}$, $\delta S^{11}, \delta S^{12}, \delta S^{22}$:

$$\delta\dot{x}_1 = \frac{1}{m}\delta p_1\,, \tag{3.48}$$

$$\delta\dot{p}_1 = -8g(G^{11} - G^{22})\delta x_1 - 8g\bar{x}_2\delta G^{12}\,, \tag{3.49}$$

$$\delta\dot{x}_2 = \frac{1}{m}\delta p_2\,, \tag{3.50}$$

$$\delta\dot{p}_2 = -8g\bar{x}_2^2 - 12g\bar{x}_2\delta G^{22}\,, \tag{3.51}$$

$$\delta\dot{G}^{11} = \frac{1}{m}\delta T^{11}\,, \tag{3.52}$$

$$\delta\dot{G}^{12} = \frac{1}{2m}\delta T^{12} + \frac{1}{2m}\delta T^{21}\,, \tag{3.53}$$

$$\delta\dot{G}^{22} = \frac{1}{m}\delta T^{22}\,, \tag{3.54}$$

$$\delta\dot{T}^{11} = -16g(G^{11} - G^{22})\delta G^{11} + \frac{2}{m}\delta S^{11}\,, \tag{3.55}$$

$$\delta\dot{T}^{12} = -16gG^{11}\bar{x}_2\delta x_1 - 16g(\bar{x}_2^2 + G^{11})\delta G^{12} + \frac{2}{m}\delta S^{12}\,, \tag{3.56}$$

$$\delta\dot{T}^{21} = -16gG^{22}\bar{x}_2\delta x_1 - 16gG^{11}\delta G^{12} + \frac{2}{m}\delta S^{12}\,, \tag{3.57}$$

$$\delta\dot{T}^{22} = -16g\bar{x}_2^2\delta G^{22} + \frac{2}{m}\delta S^{22}\,, \tag{3.58}$$

$$\delta\dot{S}^{11} = -8g(G^{11} - G^{22})\delta T^{11}\,, \tag{3.59}$$

$$\delta\dot{S}^{12} = -4g(G^{11} - G^{22})\delta T^{12} - 4g\bar{x}_2^2\delta T^{21}\,, \tag{3.60}$$

$$\delta\dot{S}^{22} = -8g\bar{x}_2^2\delta T^{22}\,, \tag{3.61}$$

To write these equations, we have used the mean-field equations for the solution with broken symmetry.

From this differential system of first order, we obtain a 14-by-14 RPA matrix. Coming back to the operator representation, with the help of relation (3.10) and its inverse, this corresponds to the 14 operators we have to include in the most general form of the excitation operator $Q_\nu^\dagger$ of the first part of this paper if we keep only bilinear operators: $a_1, a_1^\dagger, a_2, a_2^\dagger$, $a_1a_1, a_1^\dagger a_1^\dagger$, $a_2a_2, a_2^\dagger a_2^\dagger$, $a_1a_2, a_1^\dagger a_2^\dagger$, $a_1^\dagger a_2, a_1a_2^\dagger$ and $a_1^\dagger a_1, a_2^\dagger a_2$. The last two operators appear only at finite temperature (For the derivation of RPA equations at finite temperature, see ref. [18]).

The operator L_3 allows to separate these operators into two sectors: the six operators $a_1, a_1^\dagger, a_1a_2, a_1^\dagger a_2^\dagger, a_1^\dagger a_2, a_1a_2^\dagger$ corresponding to the "pion" sector and the 8 operators $a_2, a_2^\dagger, a_1a_1, a_1^\dagger a_1^\dagger, a_2a_2, a_2^\dagger a_2^\dagger$, $a_1^\dagger a_1$, $a_2^\dagger a_2$ corresponding to the "sigma"sector. By using again the correspondence between the X_i, P_i representation and the $a_i^\dagger, a_i$ representation, we deduce that the variables in our first-order differential system which correspond to the "pion" are: $\delta\bar{x}_1, \delta\bar{p}_1, \delta G^{12}, \delta T^{12}, \delta T^{21}, \delta S^{12}$, the other corresponding to the "sigma" sector. We check indeed that the RPA matrix, which we call $\mathcal{R}$, can be written in two blocks, one 6-by-6 corresponding to the "pion" and one 8-by-8 corresponding to the "sigma", the two sectors being disconnected. We can write the first-order differential system in the form:

$$\delta\dot{X} = \mathcal{R}\delta X\ , \tag{3.62}$$

where $\delta\tilde{X} = (\delta\bar{x}_1, \delta\bar{p}_1, \delta G^{12}, \delta T^{12}, \delta T^{21}, \delta S^{12}, \delta\bar{x}_2, \delta\bar{p}_2$, $\delta G^{11}, \delta G^{22}, \delta T^{11}, \delta T^{22}, \delta S^{11}, \delta S^{22})$ and

$$\mathcal{R} = \begin{pmatrix} \mathcal{M}_\pi & 0 \\ 0 & \mathcal{M}_\sigma \end{pmatrix}\ . \tag{3.63}$$

The 6-by-6 RPA matrix $\mathcal{M}_\pi$ for the "pion" sector is equal to

see eq. (3.64) *on the next page*

and the 8-by-8 RPA matrix $\mathcal{M}_\sigma$ is equal to

see eq. (3.65) *on the next page.*

We have $\det\mathcal{M}_\pi = 0$. However, before concluding about the existence of a zero mode associated to the spontaneous breakdown of the rotational symmetry, we have to eliminate the spurious modes corresponding to invariants of the TDHB evolution. At zero temperature, this will allow to reduce the $\mathcal{M}_\pi$ matrix to a 4-by-4 matrix. This is also the

$$\mathcal{M}_\pi = \begin{pmatrix} 0 & \frac{1}{m} & 0 & 0 & 0 & 0 \\ -8g(G^{11}-G^{22}) & 0 & -8g\bar{x}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2m} & \frac{1}{2m} & 0 \\ -16gG^{11}\bar{x}_2 & 0 & -16g(\bar{x}_2^2+G^{11}) & 0 & 0 & \frac{2}{m} \\ -16gG^{22}\bar{x}_2 & 0 & -16gG^{11} & 0 & 0 & \frac{2}{m} \\ 0 & 0 & 0 & -4g(G^{11}-G^{22}) & -4g\bar{x}_2^2 & 0 \end{pmatrix} \tag{3.64}$$

$$\mathcal{M}_\sigma = \begin{pmatrix} 0 & \frac{1}{m} & 0 & 0 & 0 & 0 & 0 & 0 \\ -8g\bar{x}_2^2 & 0 & 0 & -12g\bar{x}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{m} & 0 & 0 \\ 0 & 0 & -16g(G^{11}-G^{22}) & 0 & 0 & 0 & \frac{2}{m} & 0 \\ 0 & 0 & 0 & -16g\bar{x}_2^2 & 0 & 0 & 0 & \frac{2}{m} \\ 0 & 0 & 0 & 0 & -8g(G^{11}-G^{22}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -8g\bar{x}_2^2 & 0 & 0 \end{pmatrix} \tag{3.65}$$

dimension we have found in the first section in standard RPA.

At zero temperature, we have the following condition to be satisfied to have a pure state:

$$\rho(\rho+I)=0 \tag{3.66}$$

or

$$-\Xi'\tau\Xi'=\tau\ , \tag{3.67}$$

where the matrices ρ and Ξ' are given by eqs. (3.9) and (3.12) and I is the 2-by-2 unity matrix. We linearize eq. (3.67) around the HFB static solution. This gives the following conditions:

$$\sum_k G^{ik}\delta T^{jk} - \delta T^{ik}G^{kj} = 0\,, \tag{3.68}$$

$$\sum_k \delta T^{ki}S^{kj} - S^{ik}\delta T^{kj} = 0\,, \tag{3.69}$$

$$\sum_k G^{ik}\delta S^{kj} + \delta G^{ik}S^{kj} = 0\,. \tag{3.70}$$

Equations (3.68) and (3.69) give, respectively,

$$G^{11}\delta T^{21} - \delta T^{12}G^{22} = 0\,, \tag{3.71}$$

$$S^{22}\delta T^{21} - \delta T^{12}S^{11} = 0\,. \tag{3.72}$$

At the minimum at zero temperature, we have $S^{11} = 1/4G^{11}$ and $S^{22} = 1/4G^{22}$. The two previous conditions are therefore equivalent. From eq. (3.70), we obtain

$$\delta G^{12} + 4G^{11}G^{22}\delta S^{12} = 0\,, \tag{3.73}$$

$$\delta G^{11} + 4(G^{11})^2\delta S^{11} = 0\,, \tag{3.74}$$

$$\delta G^{22} + 4(G^{22})^2\delta S^{22} = 0\,. \tag{3.75}$$

The first condition is for the “pion” sector and the last two conditions are for the “sigma”sector.

Coming back to the operator representation, we have

$$2\sqrt{\omega\Omega}\left\langle \tilde{X}_1\tilde{X}_2\right\rangle + \frac{2}{\sqrt{\omega\Omega}}\left\langle \tilde{P}_1\tilde{P}_2\right\rangle = 2\left(\left\langle a_1a_2^\dagger\right\rangle + \left\langle a_1^\dagger a_2\right\rangle\right), \tag{3.76}$$

$$2\sqrt{\frac{\omega}{\Omega}}\left\langle \tilde{X}_1\tilde{P}_2 + \tilde{P}_2\tilde{X}_1\right\rangle - 2\sqrt{\frac{\Omega}{\omega}}\left\langle \tilde{X}_2\tilde{P}_1 + \tilde{P}_1\tilde{X}_2\right\rangle = 4\left(\left\langle a_1a_2^\dagger\right\rangle - \left\langle a_1^\dagger a_2\right\rangle\right)\,. \tag{3.77}$$

By using $G^{12} = \langle\tilde{X}_1\tilde{X}_2\rangle$, $S^{12} = \langle\tilde{P}_1\tilde{P}_2\rangle$, $T^{12} = 2\langle\tilde{X}_1\tilde{P}_2\rangle$, $T^{21} = 2\langle\tilde{X}_2\tilde{P}_1\rangle$, and $G^{11} = 1/\omega, G^{22} = 1/\Omega$ for the static HFB solution, we see that conditions (3.71) and (3.73) correspond to the decoupling of the operators $a_1a_2^\dagger, a_1^\dagger a_2$ we have found in the first section in standard RPA.

For the pion sector, we will therefore consider the following new variables:

$$\delta\tilde{Y} = \Big(\delta x_1, \delta p_1, \delta C = -4G^{11}G^{22}\delta G^{12} + \delta S^{12},$$
$$\delta D = G^{11}\delta T^{12} + G^{22}\delta T^{21}\,,$$
$$\delta E = \delta G^{12} + G^{22}\delta T^{12} - G^{11}\delta T^{21} + 4G^{11}G^{22}\delta S^{12}\,,$$
$$\delta F = \delta G^{12} - G^{22}\delta T^{12} + G^{11}\delta T^{21} + 4G^{11}G^{22}\delta S^{12}\Big). \tag{3.78}$$

We have

$$\delta\dot{Y} = \mathcal{M}'_\pi\,\delta Y\,. \tag{3.79}$$

From the condition to remain in the variational space of pure Gaussian states we have $\delta E = 0$, $\delta F = 0$. We therefore consider in $\mathcal{M}'_\pi$ the 4-by-4 matrix corresponding to the four coordinates $\delta x_1, \delta p_1, \delta C, \delta D$ (we notice that there are non-vanishing matrix elements in the fifth and sixth columns of $\mathcal{M}'_\pi$):

$$\mathcal{M}''_\pi = \begin{pmatrix} 0 & \frac{1}{m} & 0 & 0 \\ -8g(G^{11}-G^{22}) & 0 & \frac{32G^{11}G^{22}g\bar{x}_2}{1+16(G^{11}G^{22})^2} & 0 \\ 0 & 0 & 0 & N_1 \\ -16((G^{11})^2+(G^{22})^2)g\bar{x}_2 & 0 & N_2 & 0 \end{pmatrix}, \quad (3.80)$$

where

$$N_1 = -\frac{2}{m((G^{11})^2+(G^{22})^2)}\left[G^{11}G^{22}(G^{22}-2gm) + (G^{11})^2(G^{22}+2gm) + 2G^{22}g\bar{x}_2^2 m\right], \quad (3.81)$$

$$N_2 = -\frac{2}{m(1+(G^{11}G^{22})^2)}\Big[G^{11}+G^{22}+32(G^{11})^3G^{22}gm + 32{G^{11}}^2G^{22}gm(G^{22}+\bar{x}_2^2)\Big]. \quad (3.82)$$

We then calculate $\det \mathcal{M}''_\pi$ and, by using the two gap equations (3.44) and (3.45) for the solution with broken symmetry, we obtain

$$\det \mathcal{M}''_\pi = 0. \quad (3.83)$$

At zero temperature, we have therefore checked the existence of a zero frequency for the solution with broken symmetry in the standard RPA.

At finite temperature, the condition (3.66) becomes

$$\rho(\rho+I) = \frac{1}{4}(C-1)I, \quad (3.84)$$

where C is called the Heisenberg invariant and is a quantity conserved by the TDHB evolution: $\dot{C}=0$. In our $O(2)$ model, C has two indices: $C^{ab}, a, b = 1, 2$. The Heisenberg invariant is equal to

$$C^{ab} = \sum_c 4\langle\tilde{X}^a\tilde{X}^c\rangle\langle\tilde{P}^c\tilde{P}^b\rangle - \langle\tilde{X}^a\tilde{P}^c+\tilde{P}^c\tilde{X}^a\rangle\langle\tilde{X}^c\tilde{P}^b+\tilde{P}^b\tilde{X}^c\rangle. \quad (3.85)$$

For the static solution at finite temperature, we have $T^{ab}=0$ and

$$4G^{11}S^{11} = C^{11}, \qquad 4G^{22}S^{22} = C^{22}. \quad (3.86)$$

C^{11} and C^{22} are related to the occupation numbers for the bosons 1 and 2 according to $2n_a+1=\sqrt{C^{aa}}$. Similarly to the zero-temperature case, we linearize the condition (3.84) around the static solution characterized by G^{11}, G^{22}, S^{11} and S^{22}. It is then convenient to introduce new variables in the pion sector and we check the existence of the zero mode.

4 Conclusion

In this paper, we have applied the random-phase approximation and its extension called renormalized RPA to the quantum anharmonic oscillator with an $O(2)$ symmetry. The expression for the RPA matrix in renormalized RPA is formally very similar to the RPA matrix appearing in standard RPA at finite temperature. We focused on the existence of a zero mode among the RPA frequencies in the case where the ground state has a broken symmetry. This result is encouraging to apply renormalized RPA in the linear sigma model. We have compared also the approach with the creation and annihilation operators with the time-dependent approach and we identify the variables corresponding to the "pion" sector and those corresponding to the "sigma" sector. The numerical resolution of the self-consistent renormalized-RPA equations will be the subject of a next paper, where Hartree-Bogoliubov mean-field results, standard-RPA results and renormalized-RPA results will be compared to exact numerical results for the vacuum energy and the energy of the first-excited states in the cases of the symmetric solution and the solution with broken symmetry.

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